1.5: Linear First-Order Equations

Definition 1. A **linear first-order** differential equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{1}$$

In order to solve an equation of the form in (1), we must use the **integrating factor**

$$\rho(x) = e^{\int P(x)dx}. \tag{2}$$

<u>Solution of Linear First-Order Equations</u>

1. Begin by calculating the integrating factor $\rho(x) = e^{\int P(x)dx}$ in (2).
2. Multiply both sides of the equation by $\rho(x)$.
3. Integrate both sides of the equation.
   (a) Recognize that the left-hand side of the equation is the derivative $D_x(\rho(x)y(x))$.
   (b) Integrate the right-hand side of the equation as usual.

Assume
$\int P(x)dx = \int P(x)dx + 0$

Example 1. Solve the initial value problem

$$\frac{dy}{dx} - y = \tfrac{11}{8}e^{-x/3}, \quad y(0) = -1.$$

- $\rho(x) = e^{\int P(x)dx} = e^{-x}.$
- $\rho(x)\left[\frac{dy}{dx} - y\right] = \rho(x)\cdot\left[\frac{11}{8}e^{-x/3}\right]$

  $e^{-x}\frac{dy}{dx} - e^{-x}y = e^{\cancel{2x/3}\;-4x/3}\,\frac{11}{8}$

- $e^{-x}y = \frac{11}{8}\cdot -\frac{3}{4}e^{-4x/3} + C$

  $= -\frac{33}{32}e^{-4x/3} + C$

$\Rightarrow y = Ce^{x} - \frac{33}{32}e^{-x/3}$

$y(0) = -1$ so $C = \frac{1}{32}$

and $y = \frac{1}{32}\left[e^{x} - 33e^{-x/3}\right]$

Example 2. Find a general solution of

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x.$$

$$\frac{dy}{dx} + \frac{3x}{x^2+1} y = \frac{6x}{x^2+1}. \quad \rho(x) = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln(x^2+1)} = (x^2+1)^{3/2}$$

After multiplying by $\rho(x)$ and integrating

$$(x^2+1)^{3/2} y = \int 6x(x^2+1)^{1/2}dx = 2(x^2+1)^{3/2} + C.$$

So $y = 2 + C(x^2+1)^{-3/2}$.

Theorem 1. **(Linear First-Order Equation)** If the functions $P(x)$ and $Q(x)$ in (1) are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on I. This is just a special case of the Uniqueness + Existence Thm. from Sect. 1.3.

Exercise 1. Solve the initial value problem

$$x^2\frac{dy}{dx} = \sin x - xy, \quad y(1) = y_0.$$

$$\frac{dy}{dx} + \frac{1}{x} y = \frac{\sin x}{x^2}$$

$$\rho(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

From the formula

$$y(x) = \frac{1}{\rho(x)}\left[y_0 + \int_{x_0}^{x} \rho(t)Q(t)\,dt\right]$$

we get $y(x) = \frac{1}{x}\left[y_0 + \int_1^x \frac{\sin t}{t}\,dt\right].$

Homework. 1-25 (odd)